

Why Three Generations?

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Introduction

Anticommuting operators and fields have been known for a long time in quantum theory but only since the early 70s have anticommuting *a-numbers* played a prominent role in physics, when supersymmetry (SUSY) and BRST symmetry came to the fore. (Of course Grassmann numbers were invented much earlier in mathematics and feature prominently in differential geometry.) Since the 70s they have dominated attempts to construct a sensible, properly unified theory.

While BRST symmetry has simplified the proof of the renormalizability of QCD and assists the development of gauge theory, the applicability of SUSY has turned out to be much more problematic. Nature seems to exhibit no sign of supersymmetric partners at our present energy scales and yet this has not prevented a large numbers of researchers devoting all their time and effort into developing the subject, both in the context of ordinary field theory and string/brane extensions. Apart from the alluring beauty of SUSY, there are basically two reasons for this: it is the only consistent higher symmetry theory that combines the Poincaré Group with internal symmetries in a nontrivial way, and it naturally leads to the cancellation of quantum infinities between bosons and fermion contributions without any fine-tuning. Without jeopardising these theoretical successes of SUSY, I would like to outline a scheme which uses a-numbers in a different way agreeing with the known particle spectrum and the character of their interactions, and which may lead to the goal of a unified theory.

The principle behind this scheme is cancellation of dimensions between c-nos and a-nos. It is rather well-known that Bose and Fermi states contribute oppositely to statistical formulae, functional determinants and group properties [1]; indeed this is the fundamental reason for the cancellation between bosonic and fermionic quantum loops. Because the numbers of visible, commuting space-time coordinates x is just 4, I propose to append 5 complex anticommuting coordinates ξ to these (but will only use half of the superfield expansions in ξ , making it equivalent to four a-numbers) in order to ensure total zero-dimensionality of the universe, as it was before the big bang one presumes? I will associate these a-nos. with ‘properties’ or internal structure, giving me a theory of ‘spacetime-property’; not only will we know where and when an event occurs, but what it is! A welcome bonus of the scheme is how it naturally accommodates three generations, without invoking a particular gauge group—normally conjured out of thin air—or repetition number. Further it mimics Klein-Kaluza type models without producing ∞ numbers of modes, because a-number expansions are necessarily finite. All of these considerations suggest that a-number extensions to spacetime may be the way to go, rather than higher bosonic coordinates with their infinite excitations, compactification notwithstanding. Due

to space limitations I shall sketch out the main ideas and refer you to earlier papers which I have written with various valued collaborators [2, 3]. To my knowledge the only venture into the a-no. ‘property market’ besides ours is that of Ellicott and Toms [4].

Superfield expansions

It turns out that three or fewer ξ are not enough to accommodate three generations. Four ξ suffice, but at the price of ‘schizosymmetry’ [5], causing some discomfort because standard statistics must be interpreted unconventionally. The easiest solution is to add an extra (uncharged) a-number ξ_0 to the ξ_μ , $\mu = 1 \dots 4$, but only consider superfields with odd powers of all ξ for fermions Ψ_α , even powers for bosons Φ . This patches the statistics and leaves us with *effectively four* ξ . Taking a leaf out of SU(5) unified gauge theory, the charge and fermion number assignments are respectively taken to be $Q(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4) = 0, 1/3, 1/3, 1/3, -1$ and $F(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4) = 1, -1/3, -1/3, -1/3, 1$. Note in passing that $\text{Tr}(Q) = 0$ helps with anomaly cancellation. [The ξ are complex and given by the combination $\xi_{(1)} + i\xi_{(2)}$ of the more familiar symplectic basis $\xi_{(1,2)}$.] Any superfield is to be expanded in terms of ξ_μ and $\bar{\xi}^\nu$. The ξ and $\bar{\xi}$ being anticommuting, such expansions end at the fifth power:

$$\Phi(X) \equiv \Phi(x, \xi, \bar{\xi}) = \sum_{\text{even } r+\bar{r}} (\bar{\xi})^{\bar{r}} \phi_{(\bar{r}), (r)}(x) (\xi)^r; \Psi_\alpha(X) \equiv \Psi(x, \xi, \bar{\xi}) = \sum_{\text{odd } r+\bar{r}} (\bar{\xi})^{\bar{r}} \psi_{\alpha(\bar{r}), (r)}(x) (\xi)^r.$$

Roughly speaking we may associate labels $i = 1 - 3$ with (down) chromicity and 4 with charged leptonicity, while 0 corresponds to neutrinoity. However bear in mind that products of ξ and $\bar{\xi}$ can lead to other properties like generation number and (up)colour. This is most readily seen by drawing up the particle content in a magic 6×6 square corresponding to r, \bar{r} running from 0 to 5. Ψ, Φ are super-self-conjugate in as much as $\psi_{(r), (\bar{r})} = \psi_{(\bar{r}), (r)}^{(c)}$, thereby halving the number of components, but this still leaves too many (256) components for comfort. It pays to further halve the field by invoking anti-duality, $\psi_{(r), (\bar{r})} \sim -\psi_{(5-\bar{r}), (5-r)}$, without affecting charge assignments and cutting the number down to 120 states. Considering the source field Ψ , the resulting square contains the following varieties of up (U), down (D), charged lepton (L) and neutrinos (N), where the subscript distinguishes between repetitions. In the table below, - are duals, * are conjugates:

$r \backslash \bar{r}$	0	1	2	3	4	5
0		L_1, N_1, D_1^c		L_5^c, D_5, U_1		
1	*		$L_{2,3}, N_{2,3}, D_{2,3}^c, U_2$		L_6, D_6, U_3	
2		*		$L_4, N_4, D_{4,7}^c$		-
3	*		*		-	
4		*		*		-
5	*		*		*	

Antiduality eliminates the neutrino state corresponding to the product of all five ξ as well as doubly charged leptons associated with $\bar{\xi}^4 \xi_0 \xi_1 \xi_2 \xi_3$ and $\bar{\xi}^4 \bar{\xi}_0 \bar{\xi}_1 \bar{\xi}_2 \bar{\xi}_3$. Notice that the table has three U generations, but more generations of D, N and L are indicated. Who can be truly sure that they will not be found with larger masses? (A sterile neutrino may play a useful role anyway.) *The*

main point is that their number is finite and small. Of greater interest is the occurrence of exotic quarks with charges $4/3$ and $5/3$, tied with properties $\bar{\xi}^4 \xi_0 \xi_i$, $\bar{\xi}^4 \xi_i \xi_j$; instead of being a cause for despair one can speculate that they or the extra D may be connected with new composite hadrons like Θ^+ , as were discovered during the last two years. Until the mass matrices of these states are constructed in a realistic way all avenues are open.

The Higgs field ought to correspond to the superfield Φ with its even ξ expansion. Vacuum expectation values must be colour singlets and uncharged; a plethora of them is available: we can allow for one $\phi_{(0)(0)}$, one $\phi_{(0),(4)}$, three $\phi_{(1),(1)}$, four $\phi_{(2),(2)}$ — with the understanding that their duals are not independent. These must be able to account for *all* the quark, lepton and neutrino masses however, which is a very strong constraint!

To show how this might pan out, let me consider a simplified model in 2D space-time with two complex ξ , having the properties of ‘electronicity’ and ‘protonicity’ so their charges and fermion numbers are $Q(\xi_1, \xi_2) = -1, 1$ and $F(\xi_1, \xi_2) = 1, 1$ respectively. In this case invoke superfield self-duality to exorcise the doubly charged state $\bar{\xi}^1 \xi_2$ as well as the ‘atomic’ product property $\xi_1 \xi_2$. Assuming self-duality the expansions read

$$\Psi(\bar{\xi}, \xi) = (\bar{B}^m \xi_m + \bar{\xi}^m B_m)(1 + \bar{\xi}^n \xi_n)/2, \quad \Phi(\bar{\xi}, \xi) = (A + S \bar{\xi}^m \xi_m)(1 + \bar{\xi}^n \xi_n)/2.$$

In this model, upon integrating out the properties via $\int d^2 \xi d^2 \bar{\xi}$, for the typical Lagrangian $\mathcal{L} = h \Phi^2 \langle \Phi \rangle + g \bar{\Psi} \Psi \langle \Phi \rangle$, the nonvanishing expectation values a and s produce mass terms:

$$[h(2A^2 + S^2) + 2g\bar{B}^n B_n]a + [3hAS + g\bar{B}^n B_n]s.$$

For the above we deduce the three masses $M_{B\pm} = [3M \pm \sqrt{M^2 + 4m^2}]/2$ for bosons and $M_F = m + M$ for fermions.

In 5D the situation is much more complicated because there are many more expectation values to be accounted for; it may turn out to be quite difficult to fit all known particle masses with the nine allowable expectation values $\langle \phi \rangle$. What is certain is that the scheme is quite different from the standard simple picture with its two sets of 3×3 mass matrices (neutrinos and quarks), which has created a feeling of complacency in the particle physics community. Hints from physics in respect of new multiquark composites and our inability to provide a convincing picture of masses and V_{ij} mixings, both in the quark and neutrino sectors, are indications that not everything is understood or even 100% correct in the standard description.

Generalized Relativity

Where do the gauge fields fit into this picture? One way to introduce them would be to mimic SUSY and supergauge the massless free Lagrangian for Ψ , but without added complications of spin. [This would need to be done so that anomalies cancel and the number of fermion fields match those of the bosons.] But there is a more compelling way, which has the benefit of incorporating gravity. The idea is to devise a fermionic version of the familiar Kaluza-Klein (KK) scheme, without the need for infinite modes that arise from shrinking the usual additional bosonic coordinates. The method has been published elsewhere [3] and, with minor improvements, I would like to highlight the main points. Let $X = (x, \xi)$ stand for the spacetime-property manifold.

- If we are to use a generalized metric for X , one must introduce a fundamental length scale to match x and ξ , since properties are dimensionless. This presents an opportunity to introduce a fundamental length into physics; of course the gravity scale $\kappa = \sqrt{8\pi G_N}$ is a natural choice, particularly as it enters the spacetime sector.
- Gravity (plus gauge field products, as we shall see) fall within the $x - x$ sector, gauge fields are accommodated in the $x - \xi$ sector, and one presumes that the Higgs scalars Φ form a matrix in the $\xi - \xi$ sector.
- Gauge invariance is connected with the numbers of ξ , so the full gauge group would have to be $SU(5)$ or perhaps a subgroup.
- There is no natural place for a gravitino as the ξ are Lorentz scalar.

We envisage a real metric: $ds^2 = dx^m G_{mn} dx^n + dx^m G_m{}^\nu d\xi_\nu + d\bar{\xi}^\mu G_{\mu n} dx^n + d\bar{\xi}^\mu G_\mu{}^\nu d\xi_\nu$ where the tangent space limit corresponds to Minkowskian $G_{ab} \rightarrow I_{ab} = \eta_{ab}$, $G_\alpha{}^\beta \rightarrow I_\alpha{}^\beta = \kappa^2 \delta_\alpha{}^\beta$, multiplied at least by $(\bar{\xi}\xi)^5$ to ensure that the property integration causes no harm. Proceeding to curved space the components should contain the force fields, leading one to a ‘superbein’

$$\bar{E}_M^A = \begin{pmatrix} e_m^a & i\kappa \bar{\xi}^\gamma A_{m\gamma}{}^\alpha \\ 0 & \kappa^2 \Phi_\mu^\alpha \end{pmatrix},$$

and the metric “tensor”

$$G_{MN} = \bar{E}_M^A I_{AB} E_N^B = \begin{pmatrix} e_m^a e_{an} + \kappa^2 \bar{\xi}^\gamma A_{m\gamma}{}^\alpha A_{n\alpha}{}^\delta \xi_\delta & i\kappa \bar{\xi}^\gamma A_{m\gamma}{}^\alpha \Phi_\alpha^\nu \\ -i\kappa \Phi_\mu^\alpha A_{n\alpha}{}^\delta \xi_\delta & \kappa^2 \Phi_\mu^\alpha \Phi_\alpha^\nu \end{pmatrix}.$$

One can then show that the generalized connection contains the curl of the gauge field A , namely $F = \partial A + A \wedge A$, plus the purely gravitational connection, resulting in the generalised anti-self-dual scalar curvature,

$$R = (R^{(g)} + \kappa^2 F_{mn}{}^\nu F^{mn}{}_\nu)(1 - (\bar{\xi}\xi)^5)/4,$$

as desired, if we disregard the matrix structure of Φ and simply assume a flat expectation value $\Phi = 1$.

Gauge symmetry corresponds to the special change $\xi_\mu \rightarrow \xi'_\mu = [\exp(i\Lambda(x))]^\nu_\mu \xi_\nu$, $x \rightarrow x' = x$. Given the standard transformation law

$$G_{m\mu}(X) = \frac{\partial X'^K}{\partial x^m} \frac{\partial X'^L}{\partial \bar{\xi}^\mu} G'_{KL} = \frac{\partial \xi'_\kappa}{\partial x^m} \frac{\partial \bar{\xi}'^{\lambda}}{\partial \bar{\xi}^\mu} G'^\kappa_\lambda + \frac{\partial \bar{\xi}'^{\lambda}}{\partial \bar{\xi}^\mu} G'_{m\lambda},$$

this translates into the usual gauge variation (a matrix in property space),

$$A_m(x) = \exp(-i\Lambda(x)) [A'_m(x) - i\partial_m] \exp(i\Lambda(x)).$$

Perhaps an easier way to see this is by writing this particular metric length² in the form,

$$ds^2 = dx^m g_{mn}^{(g)} dx^n + \kappa^2 (i dx^m \bar{\xi}^\mu A_{m\mu}{}^\rho + d\bar{\xi}^\rho) (-i A_{n\rho}{}^\nu \xi_\nu dx^n + d\xi_\rho).$$

This argument does not fix what (sub)group is to be gauged in property space although one most certainly expects to handle the nonabelian colour group and the abelian electromagnetic group, to agree with physics. In fact the correct choice may well be tied to expectation values of Φ in the property sector, which I happened to set equal to unity above for simplicity.

My exposition has been necessarily sketchy due to lack of space but I trust that the general ideas have come across. A more detailed version of this scheme will be published elsewhere.

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